

Trig Integration Recurrence Equations

Sine Integration Recurrence Equations

$$\int \sin[c + dx]^m (A + B \sin[c + dx]) (a + b \sin[c + dx])^n dx \text{ when } a^2 - b^2 = 0$$

- Symmetric sine recurrence 1a: If $a^2 - b^2 = 0$, then

$$d(m+1) \int \sin[c + dx]^m (A + B \sin[c + dx]) (a + b \sin[c + dx])^n dx = \\ a A \cos[c + dx] \sin[c + dx]^{m+1} (a + b \sin[c + dx])^{n-1} +$$

$$\int \sin[c + dx]^{m+1} ((A b + a B)(m+1) - A b(n-1) + (a A n + (a A + b B)(m+1)) \sin[c + dx]) (a + b \sin[c + dx])^{n-1} dx$$

- Symmetric sine recurrence 1b: If $a^2 - b^2 = 0$, then

$$d(m+n+1) \int \sin[c + dx]^m (A + B \sin[c + dx]) (a + b \sin[c + dx])^n dx = \\ -b B \cos[c + dx] \sin[c + dx]^{m+1} (a + b \sin[c + dx])^{n-1} +$$

$$d \int \sin[c + dx]^m (a A n + (a A + b B)(m+1) + (A b + a B n + (A b + a B)(m+n)) \sin[c + dx]) (a + b \sin[c + dx])^{n-1} dx$$

- Symmetric sine recurrence 2a: If $a^2 - b^2 = 0$, then

$$a^2 d(2n+1) \int \sin[c + dx]^m (A + B \sin[c + dx]) (a + b \sin[c + dx])^n dx = \\ a(A b - a B) \cos[c + dx] \sin[c + dx]^m (a + b \sin[c + dx])^{n+1} +$$

$$d \int \sin[c + dx]^{m-1} (-m(A b - a B) + (b B n + a A(n+1) + m(a A - b B)) \sin[c + dx]) (a + b \sin[c + dx])^{n+1} dx$$

- Symmetric sine recurrence 2b: If $a^2 - b^2 = 0$, then

$$b^2 d(2n+1) \int \sin[c + dx]^m (A + B \sin[c + dx]) (a + b \sin[c + dx])^n dx = \\ -b(A b - a B) \cos[c + dx] \sin[c + dx]^{m+1} (a + b \sin[c + dx])^n +$$

$$d \int \sin[c + dx]^m (a A(2n+1) + (a A - b B)(m+1) - (A b - a B)(m+n+2) \sin[c + dx]) (a + b \sin[c + dx])^{n+1} dx$$

- Symmetric sine recurrence 3a: If $a^2 - b^2 = 0$, then

$$a d(m+n+1) \int \sin[c + dx]^m (A + B \sin[c + dx]) (a + b \sin[c + dx])^n dx = \\ -a B \cos[c + dx] \sin[c + dx]^m (a + b \sin[c + dx])^{n+1} +$$

$$d \int \sin[c + dx]^{m-1} (a B m + (b B n + a A(m+n+1)) \sin[c + dx]) (a + b \sin[c + dx])^n dx$$

- Symmetric sine recurrence 3b: If $a^2 - b^2 = 0$, then

$$a d(m+1) \int \sin[c + dx]^m (A + B \sin[c + dx]) (a + b \sin[c + dx])^n dx = \\ a A \cos[c + dx] \sin[c + dx]^{m+1} (a + b \sin[c + dx])^n +$$

$$d \int \sin[c + dx]^{m+1} (a B(m+1) - b A n + a A(m+n+2) \sin[c + dx]) (a + b \sin[c + dx])^n dx$$

$$\int \sin[c + dx]^m (A + B \sin[c + dx] + C \sin[c + dx]^2) (a + b \sin[c + dx])^n dx \text{ when } a^2 - b^2 \neq 0$$

■ Sine recurrence 1a:

$$\begin{aligned} d(m+1) \int \sin[c + dx]^m (A + B \sin[c + dx] + C \sin[c + dx]^2) (a + b \sin[c + dx])^n dx = \\ A \cos[c + dx] \sin[c + dx]^{m+1} (a + b \sin[c + dx])^n + \\ d \int \sin[c + dx]^{m+1} (aB(m+1) - bA(n+1) + (aA + (aA + bB + aC)(m+1)) \sin[c + dx] + \\ b(A(n+1) + (A+C)(m+1)) \sin[c + dx]^2) (a + b \sin[c + dx])^{n-1} dx \end{aligned}$$

■ Sine recurrence 1b:

$$\begin{aligned} d(m+n+2) \int \sin[c + dx]^m (A + B \sin[c + dx] + C \sin[c + dx]^2) (a + b \sin[c + dx])^n dx = \\ -C \cos[c + dx] \sin[c + dx]^{m+1} (a + b \sin[c + dx])^n + \\ d \int \sin[c + dx]^m (a(A(n+1) + (A+C)(m+1)) + (aB + aB + (aB + aB + bC)(m+n+1)) \sin[c + dx] + \\ (aCn + bB(m+n+2)) \sin[c + dx]^2) (a + b \sin[c + dx])^{n-1} dx \end{aligned}$$

■ Sine recurrence 2a:

$$\begin{aligned} bd(n+1)(a^2 - b^2) \int \sin[c + dx]^m (A + B \sin[c + dx] + C \sin[c + dx]^2) (a + b \sin[c + dx])^n dx = \\ -(Ab^2 - abB + a^2C) \cos[c + dx] \sin[c + dx]^m (a + b \sin[c + dx])^{n+1} + \\ d \int \sin[c + dx]^{m-1} (m(Ab^2 - abB + a^2C) + b(aA - bB + aC)(n+1) \sin[c + dx] - \\ (b(aB - aB + bC)(n+1) + (Ab^2 - abB + a^2C)(m+1)) \sin[c + dx]^2) (a + b \sin[c + dx])^{n+1} dx \end{aligned}$$

■ Sine recurrence 2b:

$$\begin{aligned} ad(n+1)(a^2 - b^2) \int \sin[c + dx]^m (A + B \sin[c + dx] + C \sin[c + dx]^2) (a + b \sin[c + dx])^n dx = \\ (Ab^2 - abB + a^2C) \cos[c + dx] \sin[c + dx]^{m+1} (a + b \sin[c + dx])^{n+1} + \\ d \int \sin[c + dx]^m (A(a^2 - b^2)(n+1) - (Ab^2 - abB + a^2C)(m+1) - \\ a(aB - aB + bC)(n+1) \sin[c + dx] + (Ab^2 - abB + a^2C)(m+n+3) \sin[c + dx]^2) (a + b \sin[c + dx])^{n+1} dx \end{aligned}$$

■ Sine recurrence 3a:

$$\begin{aligned} bd(m+n+2) \int \sin[c + dx]^m (A + B \sin[c + dx] + C \sin[c + dx]^2) (a + b \sin[c + dx])^n dx = \\ -C \cos[c + dx] \sin[c + dx]^m (a + b \sin[c + dx])^{n+1} + \\ d \int \sin[c + dx]^{m-1} (aCm + b(A + (A+C)(m+n+1)) \sin[c + dx] + (bB(n+1) + (bB - aC)(m+1)) \sin[c + dx]^2) \\ (a + b \sin[c + dx])^n dx \end{aligned}$$

■ Sine recurrence 3b:

$$\begin{aligned} ad(m+1) \int \sin[c + dx]^m (A + B \sin[c + dx] + C \sin[c + dx]^2) (a + b \sin[c + dx])^n dx = \\ A \cos[c + dx] \sin[c + dx]^{m+1} (a + b \sin[c + dx])^{n+1} + \\ d \int \sin[c + dx]^{m+1} ((aB - bA)(m+1) - bA(n+1) + a(A + (A+C)(m+1)) \sin[c + dx] + bA(m+n+3) \sin[c + dx]^2) \\ (a + b \sin[c + dx])^n dx \end{aligned}$$

$$(a + b \sin[c + d x])^n dx$$

Cosecant Integration Recurrence Equations

$$\int \text{Csc}[c + d x]^m (A + B \text{Csc}[c + d x]) (a + b \text{Csc}[c + d x])^n dx \text{ when } a^2 - b^2 = 0$$

- Symmetric cosecant recurrence 1a: If $a^2 - b^2 = 0$, then

$$\begin{aligned} & d m \int \text{Csc}[c + d x]^m (A + B \text{Csc}[c + d x]) (a + b \text{Csc}[c + d x])^n dx = \\ & \quad a A \cos[c + d x] \text{Csc}[c + d x]^{m+1} (a + b \text{Csc}[c + d x])^{n-1} + \\ & d \int \text{Csc}[c + d x]^{m+1} ((A b + a B) m - A b (n-1) + (a A n + (a A + b B) m) \text{Csc}[c + d x]) (a + b \text{Csc}[c + d x])^{n-1} dx \end{aligned}$$

- Symmetric cosecant recurrence 1b: If $a^2 - b^2 = 0$, then

$$\begin{aligned} & d (m+n) \int \text{Csc}[c + d x]^m (A + B \text{Csc}[c + d x]) (a + b \text{Csc}[c + d x])^n dx = \\ & \quad -b B \cos[c + d x] \text{Csc}[c + d x]^{m+1} (a + b \text{Csc}[c + d x])^{n-1} + \\ & d \int \text{Csc}[c + d x]^m (a A n + (a A + b B) m + (A b + a B n + (A b + a B) (m+n-1)) \text{Csc}[c + d x]) (a + b \text{Csc}[c + d x])^{n-1} dx \end{aligned}$$

- Symmetric cosecant recurrence 2a: If $a^2 - b^2 = 0$, then

$$\begin{aligned} & a^2 d (2n+1) \int \text{Csc}[c + d x]^m (A + B \text{Csc}[c + d x]) (a + b \text{Csc}[c + d x])^n dx = \\ & \quad a (A b - a B) \cos[c + d x] \text{Csc}[c + d x]^m (a + b \text{Csc}[c + d x])^n + \\ & \quad d \int \text{Csc}[c + d x]^{m-1} \\ & \quad (- (A b - a B) (m-1) + (b B n + a A (n+1) + (a A - b B) (m-1)) \text{Csc}[c + d x]) (a + b \text{Csc}[c + d x])^{n+1} dx \end{aligned}$$

- Symmetric cosecant recurrence 2b: If $a^2 - b^2 = 0$, then

$$\begin{aligned} & b^2 d (2n+1) \int \text{Csc}[c + d x]^m (A + B \text{Csc}[c + d x]) (a + b \text{Csc}[c + d x])^n dx = \\ & \quad -b (A b - a B) \cos[c + d x] \text{Csc}[c + d x]^{m+1} (a + b \text{Csc}[c + d x])^n + \\ & d \int \text{Csc}[c + d x]^m (a A (2n+1) + (a A - b B) m - (A b - a B) (m+n+1) \text{Csc}[c + d x]) (a + b \text{Csc}[c + d x])^{n+1} dx \end{aligned}$$

- Symmetric cosecant recurrence 3a: If $a^2 - b^2 = 0$, then

$$\begin{aligned} & a d (m+n) \int \text{Csc}[c + d x]^m (A + B \text{Csc}[c + d x]) (a + b \text{Csc}[c + d x])^n dx = \\ & \quad -a B \cos[c + d x] \text{Csc}[c + d x]^m (a + b \text{Csc}[c + d x])^n + \\ & d \int \text{Csc}[c + d x]^{m-1} (a B (m-1) + (b B n + a A (m+n)) \text{Csc}[c + d x]) (a + b \text{Csc}[c + d x])^n dx \end{aligned}$$

- Symmetric cosecant recurrence 3b: If $a^2 - b^2 = 0$, then

$$\begin{aligned} & a d m \int \text{Csc}[c + d x]^m (A + B \text{Csc}[c + d x]) (a + b \text{Csc}[c + d x])^n dx = \\ & \quad a A \cos[c + d x] \text{Csc}[c + d x]^{m+1} (a + b \text{Csc}[c + d x])^n + \\ & d \int \text{Csc}[c + d x]^{m+1} (a B m - b A n + a A (m+n+1) \text{Csc}[c + d x]) (a + b \text{Csc}[c + d x])^n dx \end{aligned}$$

$$\int \text{Csc}[c + dx]^m (A + B \text{Csc}[c + dx] + C \text{Csc}[c + dx]^2) (a + b \text{Csc}[c + dx])^n dx \text{ when } a^2 - b^2 \neq 0$$

■ Cosecant recurrence 1a:

$$\begin{aligned} & d m \int \text{Csc}[c + dx]^m (A + B \text{Csc}[c + dx] + C \text{Csc}[c + dx]^2) (a + b \text{Csc}[c + dx])^n dx = \\ & \quad A \cos[c + dx] \text{Csc}[c + dx]^{m+1} (a + b \text{Csc}[c + dx])^n + \\ & d \int \text{Csc}[c + dx]^{m+1} (a B m - b A n + (a A + (a A + a C + b B) m) \text{Csc}[c + dx] + b (A (n + 1) + (A + C) m) \text{Csc}[c + dx]^2) \\ & \quad (a + b \text{Csc}[c + dx])^{n-1} dx \end{aligned}$$

■ Cosecant recurrence 1b:

$$\begin{aligned} & d (m + n + 1) \int \text{Csc}[c + dx]^m (A + B \text{Csc}[c + dx] + C \text{Csc}[c + dx]^2) (a + b \text{Csc}[c + dx])^n dx = \\ & \quad -C \cos[c + dx] \text{Csc}[c + dx]^{m+1} (a + b \text{Csc}[c + dx])^n + \\ & d \int \text{Csc}[c + dx]^m (a (A (n + 1) + (A + C) m) + (b A + a B + (b A + a B + b C) (m + n)) \text{Csc}[c + dx] + \\ & \quad (a C n + b B (m + n + 1)) \text{Csc}[c + dx]^2) (a + b \text{Csc}[c + dx])^{n-1} dx \end{aligned}$$

■ Cosecant recurrence 2a:

$$\begin{aligned} & b d (n + 1) (a^2 - b^2) \int \text{Csc}[c + dx]^m (A + B \text{Csc}[c + dx] + C \text{Csc}[c + dx]^2) (a + b \text{Csc}[c + dx])^n dx = \\ & \quad - (A b^2 - a b B + a^2 C) \cos[c + dx] \text{Csc}[c + dx]^m (a + b \text{Csc}[c + dx])^{n+1} + \\ & d \int \text{Csc}[c + dx]^{m-1} ((A b^2 - a b B + a^2 C) (m - 1) + b (a A - b B + a C) (n + 1) \text{Csc}[c + dx] - \\ & \quad ((A b^2 - a b B + b^2 C) (n + 1) + (A b^2 - a b B + a^2 C) m) \text{Csc}[c + dx]^2) (a + b \text{Csc}[c + dx])^{n+1} dx \end{aligned}$$

■ Cosecant recurrence 2b:

$$\begin{aligned} & a d (n + 1) (a^2 - b^2) \int \text{Csc}[c + dx]^m (A + B \text{Csc}[c + dx] + C \text{Csc}[c + dx]^2) (a + b \text{Csc}[c + dx])^n dx = \\ & \quad (A b^2 - a b B + a^2 C) \cos[c + dx] \text{Csc}[c + dx]^{m+1} (a + b \text{Csc}[c + dx])^{n+1} + \\ & d \int \text{Csc}[c + dx]^m (A (a^2 - b^2) (n + 1) - (A b^2 - a b B + a^2 C) m - \\ & a (b A - a B + b C) (n + 1) \text{Csc}[c + dx] + (A b^2 - a b B + a^2 C) (m + n + 2) \text{Csc}[c + dx]^2) (a + b \text{Csc}[c + dx])^{n+1} dx \end{aligned}$$

■ Cosecant recurrence 3a:

$$\begin{aligned} & b d (m + n + 1) \int \text{Csc}[c + dx]^m (A + B \text{Csc}[c + dx] + C \text{Csc}[c + dx]^2) (a + b \text{Csc}[c + dx])^n dx = \\ & \quad -C \cos[c + dx] \text{Csc}[c + dx]^m (a + b \text{Csc}[c + dx])^{n+1} + \\ & d \int \text{Csc}[c + dx]^{m-1} (a C (m - 1) + b (A + (A + C) (m + n)) \text{Csc}[c + dx] + (b B (n + 1) + (b B - a C) m) \text{Csc}[c + dx]^2) \\ & \quad (a + b \text{Csc}[c + dx])^n dx \end{aligned}$$

■ Cosecant recurrence 3b:

$$\begin{aligned} & a d m \int \text{Csc}[c + dx]^m (A + B \text{Csc}[c + dx] + C \text{Csc}[c + dx]^2) (a + b \text{Csc}[c + dx])^n dx = \\ & \quad A \cos[c + dx] \text{Csc}[c + dx]^{m+1} (a + b \text{Csc}[c + dx])^{n+1} + \\ & d \int \text{Csc}[c + dx]^{m+1} \\ & \quad ((a B - b A) m - b A (n + 1) + a (A + (A + C) m) \text{Csc}[c + dx] + b A (m + n + 2) \text{Csc}[c + dx]^2) (a + b \text{Csc}[c + dx])^n dx \end{aligned}$$

Tangent Integration Recurrence Equations

$$\int \tan[c + d x]^m (A + B \tan[c + d x]) (a + b \tan[c + d x])^n dx \text{ when } a^2 + b^2 = 0$$

- Symmetric tangent recurrence 1a (10): If $a^2 + b^2 = 0$, then

$$\begin{aligned} d(m+1) \int \tan[c + d x]^m (a + b \tan[c + d x])^n (A + B \tan[c + d x]) dx = \\ A a \tan[c + d x]^{m+1} (a + b \tan[c + d x])^{n-1} - \\ d \int \tan[c + d x]^{m+1} (A b(n-1) - (A b + B a)(m+1) + (A a(m+n) - B b(m+1)) \tan[c + d x]) (a + b \tan[c + d x])^{n-1} dx \end{aligned}$$

- Symmetric tangent recurrence 1b (9): If $a^2 + b^2 = 0$, then

$$\begin{aligned} d(m+n) \int \tan[c + d x]^m (a + b \tan[c + d x])^n (A + B \tan[c + d x]) dx = \\ B b \tan[c + d x]^{m+1} (a + b \tan[c + d x])^{n-1} + \\ d \int \tan[c + d x]^m (A a(n+m) - B b(m+1) + (B a(n-1) + (A b + B a)(m+n)) \tan[c + d x]) (a + b \tan[c + d x])^{n-1} dx \end{aligned}$$

- Symmetric tangent recurrence 2a (7): If $a^2 + b^2 = 0$, then

$$\begin{aligned} 2 a^2 n d \int \tan[c + d x]^m (a + b \tan[c + d x])^n (A + B \tan[c + d x]) dx = \\ -a(A b - B a) \tan[c + d x]^m (a + b \tan[c + d x])^n + \\ d \int \tan[c + d x]^{m-1} ((A b - B a)m + (B b(m-n) + A a(m+n)) \tan[c + d x]) (a + b \tan[c + d x])^{n+1} dx \end{aligned}$$

- Symmetric tangent recurrence 2b (12): If $a^2 + b^2 = 0$, then

$$\begin{aligned} 2 a^2 n d \int \tan[c + d x]^m (a + b \tan[c + d x])^n (A + B \tan[c + d x]) dx = \\ -a(a A + b B) \tan[c + d x]^{1+m} (a + b \tan[c + d x])^n + \\ d \int \tan[c + d x]^m (b B(m+1) + a A(m+2n+1) - (A b - a B)(m+n+1) \tan[c + d x]) (a + b \tan[c + d x])^{n+1} dx \end{aligned}$$

- Symmetric tangent recurrence 3a (8): If $a^2 + b^2 = 0$, then

$$\begin{aligned} a d(m+n) \int \tan[c + d x]^m (a + b \tan[c + d x])^n (A + B \tan[c + d x]) dx = \\ a B \tan[c + d x]^m (a + b \tan[c + d x])^n + \\ d \int \tan[c + d x]^{m-1} (-(B a m) + (A a m + (A a - B b)n) \tan[c + d x]) (a + b \tan[c + d x])^n dx \end{aligned}$$

- Symmetric tangent recurrence 3b (11): If $a^2 + b^2 = 0$, then

$$\begin{aligned} a d(m+1) \int \tan[c + d x]^m (a + b \tan[c + d x])^n (A + B \tan[c + d x]) dx = \\ a A \tan[c + d x]^{m+1} (a + b \tan[c + d x])^n - \\ d \int \tan[c + d x]^{m+1} (A b n - B a(m+1) + A a(m+n+1) \tan[c + d x]) (a + b \tan[c + d x])^n dx \end{aligned}$$

$$\int \tan[c + dx]^m (A + B \tan[c + dx] + C \tan[c + dx]^2) (a + b \tan[c + dx])^n dx \text{ when } a^2 + b^2 \neq 0$$

■ Tangent recurrence 1a:

$$\begin{aligned} d(m+1) \int \tan[c + dx]^m (A + B \tan[c + dx] + C \tan[c + dx]^2) (a + b \tan[c + dx])^n dx = \\ A \tan[c + dx]^{m+1} (a + b \tan[c + dx])^n + \\ d \int \tan[c + dx]^{m+1} (A B(m+1) - A b n + (b B - a(A - C)) (m+1) \tan[c + dx] + \\ b(C(m+1) - A(m+n+1)) \tan[c + dx]^2) (a + b \tan[c + dx])^{n-1} dx \end{aligned}$$

■ Tangent recurrence 1b:

$$\begin{aligned} d(m+n+1) \int \tan[c + dx]^m (A + B \tan[c + dx] + C \tan[c + dx]^2) (a + b \tan[c + dx])^n dx = \\ C \tan[c + dx]^{m+1} (a + b \tan[c + dx])^n + \\ d \int \tan[c + dx]^m ((A a(m+n+1) - C(m+1)a) + \\ (a B + b(A - C)) (m+n+1) \tan[c + dx] + (a C n + b B(m+n+1)) \tan[c + dx]^2) (a + b \tan[c + dx])^{n-1} dx \end{aligned}$$

■ Tangent recurrence 2a:

$$\begin{aligned} b d(n+1) (a^2 + b^2) \int \tan[c + dx]^m (A + B \tan[c + dx] + C \tan[c + dx]^2) (a + b \tan[c + dx])^n dx = \\ (A b^2 - a b B + a^2 C) \tan[c + dx]^m (a + b \tan[c + dx])^{n+1} + \\ d \int \tan[c + dx]^{m-1} (- (A b^2 - a b B + a^2 C) m + b(b B + a(A - C)) (n+1) \tan[c + dx] - \\ (b(A b - a B) (m+n+1) - C(-a^2 m + b^2(n+1))) \tan[c + dx]^2) (a + b \tan[c + dx])^{n+1} dx \end{aligned}$$

■ Tangent recurrence 2b:

$$\begin{aligned} a d(n+1) (a^2 + b^2) \int \tan[c + dx]^m (A + B \tan[c + dx] + C \tan[c + dx]^2) (a + b \tan[c + dx])^n dx = \\ - (A b^2 - a b B + a^2 C) \tan[c + dx]^{m+1} (a + b \tan[c + dx])^{n+1} + \\ d \int \tan[c + dx]^m (-a(b B - a C) (m+1) + A(a^2(n+1) + b^2(m+n+2))) + \\ a(a B - b(A - C)) (n+1) \tan[c + dx] + (A b^2 - a b B + a^2 C) (m+n+2) \tan[c + dx]^2) (a + b \tan[c + dx])^{n+1} dx \end{aligned}$$

■ Tangent recurrence 3a:

$$\begin{aligned} b d(m+n+1) \int \tan[c + dx]^m (A + B \tan[c + dx] + C \tan[c + dx]^2) (a + b \tan[c + dx])^n dx = \\ C \tan[c + dx]^m (a + b \tan[c + dx])^{n+1} - \\ d \int \tan[c + dx]^{m-1} \\ (a C m - b(A - C) (m+n+1) \tan[c + dx] + (a C m - b B(m+n+1)) \tan[c + dx]^2) (a + b \tan[c + dx])^n dx \end{aligned}$$

■ Tangent recurrence 3b:

$$\begin{aligned} a d(m+1) \int \tan[c + dx]^m (A + B \tan[c + dx] + C \tan[c + dx]^2) (a + b \tan[c + dx])^n dx = \\ A \tan[c + dx]^{m+1} (a + b \tan[c + dx])^{n+1} + \\ d \int \tan[c + dx]^{m+1} \\ (a B(m+1) - A b(m+n+2) - a(A - C) (m+1) \tan[c + dx] - A b(m+n+2) \tan[c + dx]^2) (a + b \tan[c + dx])^n dx \end{aligned}$$